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Article

A Family of New Generating Functions for the Chebyshev Polynomials, Based on Works by Laplace, Lagrange and Euler

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Abstract: Analyzing, developing and exploiting results obtained by Laplace in 1785 on the Fourier-series expansion of the function $(1 - 2\alpha\cos\theta + \alpha^2)^{-s}$, we obtain a family of new expansions and generating functions for the Chebyshev polynomials. A relation between the generating functions of the Chebyshev polynomials T_n and the Gegenbauer polynomials $C_n^{(2)}$ is given.

Keywords: orthogonal polynomials; expansions; generating functions; Legendre polynomials; Chebyshev polynomials; Gegenbauer polynomials

MSC: 33D45; 42C05; 01A50; 01A55; 33C45; 41A10; 42C10

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1. Preamble

There are various ways for introducing families of orthogonal polynomials. First of all, they satisfy an orthogonality condition in an interval with respect to a positive measure. They are also solutions of a second-order linear differential equation (equivalent to an eigenvalue problem); they satisfy a three-term recurrence relationship with certain conditions on its form and its coefficients, which ensure, by the Shohat–Favard theorem, that they are orthogonal; they verify the Christoffel–Darboux identity, which implies the recurrence relation; they have a structure relation; and they are given by the Rodrigues formula. For the so-called *classical* orthogonal families (that is the Jacobi, Bessel, Laguerre and Hermite polynomials), they can be characterized by the fact that their first derivatives also form a family of orthogonal polynomials. But the most common way of introducing them, which is also the historical way, is by using their *generating function*—that is, denoting generically by P_n these orthogonal polynomials, a formal relation of the form

$$G(x,\alpha) = \sum_{n=0}^{\infty} a_n P_n(x)\alpha^n.$$

Although it implicitly appeared for solving difference equations in the book of Abraham de Moivre (1667–1754) published in 1718 [1], the concept of generating function was introduced by Laplace in 1779 [2] and reproduced in the first book of his treatise on the analytic theory of probability [3] (Chap. I, pp. 7–48).

All the properties of a family of orthogonal polynomials can be obtained from the generating functions, and this is why it is useful to have several of them at one's disposal. This topic gave rise to a waste literature, which shows its primary importance in the domain. Generating functions for classical orthogonal polynomials were studied by the English mathematician George Neville Watson (1886–1965) in four papers [4–7]. Unusual generating functions were given by Fred Brafman (1923–1959) [8–10], whose work is reviewed in [11]. On generating functions, see [12–17]. Their history in probability theory is described in [18].

In this paper, we develop and exploit a result due to Laplace, which leads us to a new *generating function* for the Chebyshev polynomials of the first kind derived in Section 6. In the second section, we provide the motivation for this work. We explain how generating functions guided Legendre to the discovery of his orthogonal polynomials in a work on the attraction of celestial bodies. He was rapidly followed by Laplace, who was studying the same problem and also obtained the Legendre polynomials. In the third section, we analyze the work presented by Laplace in another paper on Jupiter and Saturn. In it, he generalized the function that Legendre and himself had used in their previous works by replacing the exponent $1/2$ with s in the generating function, and he deduced some properties. We check all the results that he provided. In the fourth section, we discuss some results due to Lagrange and Euler, who had also considered this generalized function but without proceeding to its complete analysis. The work of Laplace is developed in the section that follows. The calculations of Laplace are verified and continued, thus leading to new results on the extended generating function. In the last section, we exploit these results. The Fourier expansion of the generalized generating function given by Laplace is related to the Chebyshev polynomials, for which we obtain new generating functions. A relation to the generating function of Gegenbauer polynomials is also established. Our conclusion ends the paper.

2. Motivation

If one knows two sides a and b of a triangle and the angle γ between them, the third side c is given by the *law of cosines*, known since a long time ago:

$$c = \sqrt{a^2 - 2ab\cos\gamma + b^2}.$$

At the end of the 18th century, scientists were interested in the shape of the Earth and the attraction of celestial bodies. On Wednesday 22 January 1783, Adrien Marie Legendre (1752–1833) began to read at the French Academy of Sciences a memoir on the attraction of spheroids [19].

He denoted by C the center of a spheroid all of whose sections are elliptical, and by S a point outside it but on one of its axes and at a distance $CS = r$. He set $CM = z$ and, for angles, $\widehat{BCS} = \omega, \widehat{BCM} = \psi, \widehat{MPQ} = \theta$ and $\widehat{MCS} = \mu$, from which he obtained $(MS)^2 = r^2 - 2rz\cos\mu + z^2$ and $\cos\mu = \cos\omega\cos\psi + \sin\omega\sin\psi\cos\theta$. He gave the expression of the attraction (P) on S of a particule of mass dM, located at the point denoted by M above, in the plane of the meridian, along SC and perpendicular to it. The formula for (P) is

$$(P) = \int \frac{(r - z\cos\mu)\, dM}{(r^2 - 2rz\cos\mu + z^2)^{1/2}}.$$

For computing (P) he used the following series expansion:

$$\int \frac{(r - z\cos\mu)z^2\, dz}{(r^2 - 2rz\cos\mu + z^2)^{1/2}} = \int \frac{z^2\, dz}{r^2}\left[1 + 3A\frac{z^2}{r^2} + 5B\frac{z^4}{r^4} + 7C\frac{z^6}{r^6} + 9D\frac{z^8}{r^8} + \cdots\right],$$

where the coefficients A, B, C, D are the following functions of $\cos\mu$:

$$\begin{aligned}
A &= \frac{3}{2}\cos^2\mu - \frac{1}{2} \\
B &= \frac{5\cdot 7}{2\cdot 4}\cos^4\mu - \frac{3\cdot 5}{2\cdot 4}\cdot 2\cos^2\mu + \frac{1\cdot 3}{2\cdot 4} \\
C &= \frac{7\cdot 9\cdot 11}{2\cdot 4\cdot 6}\cos^6\mu - \frac{5\cdot 7\cdot 9}{2\cdot 4\cdot 6}\cdot 3\cos^4\mu + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\cdot 3\cos^2\mu - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} \\
D &= \frac{9\cdot 11\cdot 13\cdot 15}{2\cdot 4\cdot 6\cdot 8}\cos^8\mu - \frac{7\cdot 9\cdot 11\cdot 13}{2\cdot 4\cdot 6\cdot 8}\cdot 4\cos^6\mu + \frac{5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8}\cdot 6\cos^4\mu \\
&\quad - \frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\cdot 4\cos^2\mu + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}.
\end{aligned}$$

These functions do not contain odd powers of $\cos\mu$ because Legendre assumed that his attracting body was symmetrical with respect to the equator and rejected them.

Replacing $\cos\mu$ by x, these coefficients are the first Legendre polynomials of degrees $2, 4, 6, 8$. This is what Legendre did in [20], where he denoted A by X^I, B by X^{II}, and so on. He explained that these polynomials come out of the expansion of $(1-2xz+z^2)^{-1/2}$, and that it exactly holds that

$$\frac{1/2}{\sqrt{1-2xz+z^2}}+\frac{1/2}{\sqrt{1+2xz+z^2}}=1+X^Iz^2+X^{II}z^4+X^{III}z^6+\cdots.$$

This is the *generating function* of the Legendre polynomials of even degrees. A list of theorems followed with, in particular, their orthogonality property. Finally, in [21], Legendre considered the polynomials of even and odd degrees, and he gave their generating function.

In 1784, Laplace presented to the French Academy of Sciences a paper where he considered the series expansion of the attractions for arbitrary spheroids [22]. Let $r=\sqrt{a^2+b^2+c^2}$ be the distance from the origin to the attracted point located at the interior of the spheroid; θ be the angle between the radius r and the x axis; and ω be the angle between the plane passing by the x axis and the attracted point and a plane passing by the x and y axis. Laplace obtained

$$a=r\cos\theta,\quad b=r\sin\theta\cos\omega,\quad c=r\sin\theta\sin\omega.$$

Now, let $R=\sqrt{x^2+y^2+z^2}$ be the distance of the molecule to the origin, and let θ' and ω' be the angles similar to those of this molecule. It holds that

$$x=R\cos\theta',\quad y=R\sin\theta'\cos\omega',\quad z=R\sin\theta'\sin\omega'.$$

The distance of the molecule to the attracted point becomes

$$1/T=\sqrt{r^2-2rR[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\omega-\omega')]+R^2},$$

and the potential V is given by

$$V=\int TR^2\,\partial R\,\partial\omega'\,\partial\theta'\ \sin\theta',$$

where the integration with respect to R is taken from 0 to its value on the surface of the spheroid, that of ω' from 0 to 2π and that of θ' from 0 to π. The potential V satisfies an equation that is the well-known *Laplace equation* in polar coordinates. The function T satisfies the Laplace equation, and its series expansion is

$$T=Q^{(0)}/r+Q^{(1)}R/r^2+Q^{(2)}R^2/r^3+\cdots$$

The $Q^{(i)}$s are the *Laplace coefficients*, and are nothing else than the Legendre polynomials after replacing x by $\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\omega-\omega')$.

Thus, we see that the law of cosines, which leads to the functions $(1-2\alpha\cos\theta+\alpha^2)^{-1/2}$ or $(1-2\alpha x+\alpha^2)^{-1/2}$, played a fundamental role in the birth of orthogonal polynomials via their generating function. The authors of this paper are preparing a book on a chronological history of the birth and early developments of orthogonal polynomials.

3. A Work by Laplace

In a 1785 paper devoted to the theory of Jupiter and Saturn [23] (p. 124ff.), Pierre-Simon Laplace (23 March 1749–5 March 1827) was brought to consider the Fourier expansion of the function

$$(1-2\alpha\cos\theta+\alpha^2)^{-s}=b_s^{(0)}/2+b_s^{(1)}\cos\theta+b_s^{(2)}\cos 2\theta+b_s^{(3)}\cos 3\theta+\cdots, \tag{1}$$

where the $b_s^{(i)}$ are functions of α and s. They are, in fact, given by

$$b_s^{(i)}=\frac{1}{\pi}\int_0^{\pi}(1-2\alpha\cos\theta+\alpha^2)^{-s}\cos i\theta\,d\theta,$$

and, using the notation $\zeta=e^{i\theta}$, it holds that $b_s^{(i)}=b_s^{(-i)}$,

$$1-2\alpha\cos\theta+\alpha^2=1+\alpha^2-\alpha(\zeta+\zeta^{-1})=(1-\alpha\zeta)(1-\alpha\zeta^{-1}),$$

and the preceding expansion becomes $(1-2\alpha\cos\theta+\alpha^2)^{-s}=\sum_{i=-\infty}^{\infty}b_s^{(i)}\zeta^i$ [24].

Since Laplace did not give the detail of his calculations, we checked all the results that he provided.

Taking the logarithmic derivative of both members with respect to θ, Laplace obtained

$$\frac{-2s\alpha\sin\theta}{1-2\alpha\cos\theta+\alpha^2}=\frac{-b_s^{(1)}\sin\theta-2b_s^{(2)}\sin 2\theta-\cdots}{b_s^{(0)}/2+b_s^{(1)}\cos\theta+b_s^{(2)}\cos 2\theta+\cdots}.$$

Cross-multiplying, using the formulæ

$$\begin{aligned}2\cos a\cos b &= \cos(a+b)+\cos(a-b)\\ 2\sin a\cos b &= \sin(a+b)+\sin(a-b)\\ 2\sin a\sin b &= \cos(a-b)-\cos(a+b),\end{aligned}$$

and comparing the coefficients of the terms in cosines, he found the three-term recurrence relation,

$$(i-s)\alpha b_s^{(i)}=(i-1)(1+\alpha^2)b_s^{(i-1)}-(i+s-2)\alpha b_s^{(i-2)}, \tag{2}$$

and added that all $b_s^{(i)}$ can be computed from this relation if $b_s^{(0)}$ and $b_s^{(1)}$ are known.

Replacing s by $s+1$ in (1), multiplying it by $1-2\alpha\cos\theta+\alpha^2$, replacing $(1-2\alpha\cos\theta+\alpha^2)^{-s}$ by its series expansion, Laplace obtained

$$b_s^{(0)}/2+b_s^{(1)}\cos\theta+b_s^{(2)}\cos 2\theta+\cdots=(1-2\alpha\cos\theta+\alpha^2)(b_{s+1}^{(0)}/2+b_{s+1}^{(1)}\cos\theta+\cdots).$$

Then, comparing similar cosines, Laplace found that

$$b_s^{(i)}=(1+\alpha^2)b_{s+1}^{(i)}-\alpha b_{s+1}^{(i-1)}-\alpha b_{s+1}^{(i+1)}. \tag{3}$$

Formula (2) gives, by replacing s by $s+1$ and i by $i+1$,

$$(i-s)\alpha b_{s+1}^{(i+1)}=i(1+\alpha^2)b_{s+1}^{(i)}-(i+s)\alpha b_{s+1}^{(i-1)}. \tag{4}$$

Multiplying relation (3) by $(i-s)$ and using the preceding relation, it becomes

$$(i-s)b_s^{(i)}=2s\alpha b_{s+1}^{(i-1)}-s(1+\alpha^2)b_{s+1}^{(i)}. \tag{5}$$

Changing i into $i+1$, it holds that

$$(i-s+1)b_s^{(i+1)}=2s\alpha b_{s+1}^{(i)}-s(1+\alpha^2)b_{s+1}^{(i+1)}.$$

Multiplying this relation by $(i-s)\alpha$ and substituting $b_{s+1}^{(i+1)}$ by its expression given by (4), Laplace obtained

$$(i-s)(i-s+1)\alpha b_s^{(i+1)} = s(i+s)\alpha(1+\alpha^2)b_{s+1}^{(i-1)} + s[2(i-s)\alpha^2 - i(1+\alpha^2)^2]b_{s+1}^{(i)}. \quad (6)$$

Laplace then wrote that *ces deux expressions de* $b_s^{(i)}$ *et de* $b_s^{(i+1)}$ *donnent*... ("these two expressions of $b_s^{(i)}$ and of $b_s^{(i+1)}$ give..."), which meant that he eliminated $b_{s+1}^{(i-1)}$ by multiplying (5) by $(i+s)(1+\alpha^2)$, (6) by 2 and subtracting, and he obtained

$$(1-\alpha^2)^2 s b_{s+1}^{(i)} = (i+s)(1+\alpha^2)b_s^{(i)} - 2(i-s+1)\alpha b_s^{(i+1)}.$$

Changing i into $-i$ and, since $b_s^{(i)} = b_s^{(-i)}$, he finally obtained

$$(1-\alpha^2)^2 s b_{s+1}^{(i)} = (s-i)(1+\alpha^2)b_s^{(i)} + 2(i+s-1)\alpha b_s^{(i-1)}, \quad (7)$$

which allows the computation of the coefficients $b_{s+1}^{(i)}$ if the $b_s^{(i)}$ are known.

Setting $\lambda = 1 - 2\alpha\cos\theta + \alpha^2$, he obtained, by differentiating λ^{-s} with respect to α, and after some calculations,

$$\frac{\partial b_s^{(i)}}{\partial \alpha} = \frac{i + (i+2s)\alpha^2}{\alpha(1-\alpha^2)} b_s^{(i)} - \frac{2(i-s+1)}{1-\alpha^2} b_s^{(i+1)}.$$

Differentiating again, Laplace explained that the $b_s^{(i)}$ and their successive derivatives can all be obtained from $b_s^{(0)}$ and $b_s^{(1)}$. For obtaining these two expressions, he used the identity $\lambda^{-s} = (1-\alpha e^{i\theta})^{-s}(1-\alpha e^{-i\theta})^{-s}$, developed each member into a series, multiplied them together and found, for $i=0$ and $i=1$, that

$$b_s^{(0)} = 2\left[1 + s^2\alpha^2 + \left(\frac{s(s+1)}{2!}\right)^2\alpha^4 + \left(\frac{s(s+1)(s+2)}{3!}\right)^2\alpha^6 + \cdots\right], \quad (8)$$

$$b_s^{(1)} = 2\alpha\left[s + \frac{s}{1}\frac{s(s+1)}{2!}\alpha^2 + \frac{s(s+1)}{2!}\frac{s(s+1)(s+2)}{3!}\alpha^4 + \cdots\right]. \quad (9)$$

In the theory of planets, $s = 1/2$ and these series do not converge quickly if α is not quite small. On the contrary, their convergence is fast for $s = -1/2$ if $\alpha^2 < 1/2$, which is the case in the theory of Jupiter and Saturn, since $\alpha^2 < 1/3$. Thus, having computed $b_{-1/2}^{(0)}$ and $b_{-1/2}^{(1)}$ by these series, Laplace obtained $b_{1/2}^{(0)} = b_{-1/2+1}^{(0)}$ and $b_{1/2}^{(1)} = b_{-1/2+1}^{(1)}$ by his Formula (7) linking the $b_{s+1}^{(i)}$ with the $b_s^{(i)}$.

Laplace reproduced these results in his *Traité de mécanique céleste* [25], Tome I, Livre II, no. 49, pages 267ff., in Chapter VI, entitled *Seconde approximation des mouvements célestes, ou théorie de leur perturbation* ("Second approximation of celestial movements, or theory of their disturbance").

4. A Work by Lagrange

The expansion of $(1-2\alpha\cos\theta+\alpha^2)^{-s}$ into a cosine series was already given by Lagrange in 1762 [26] (Œuvres, vol. 1, page 620) in a section also devoted to Jupiter and Saturn. He set

$$P = 1 + s\alpha\cos\theta + \frac{s(s+1)}{2!}\alpha^2\cos 2\theta + \frac{s(s+1)(s+2)}{3!}\alpha^3\cos 3\theta + \cdots$$

$$Q = 1 + s\alpha\sin\theta + \frac{s(s+1)}{2!}\alpha^2\sin 2\theta + \frac{s(s+1)(s+2)}{3!}\alpha^3\sin 3\theta + \cdots,$$

and, since

$$\begin{aligned} P + iQ &= [1 - \alpha(\cos\theta + i\sin\theta)]^{-s} \\ P - iQ &= [1 - \alpha(\cos\theta - i\sin\theta)]^{-s}, \end{aligned}$$

Lagrange obtained $(1 - 2\alpha\cos\theta + \alpha^2)^{-s} = P^2 + Q^2$. Using the relation $\cos m\theta \cos n\theta + \sin m\theta \sin n\theta = \cos(m-n)\theta$, he arrived at

$$(1 - 2\alpha\cos\theta + \alpha^2)^{-s} = \mathcal{A} + \mathcal{B}\cos\theta + \mathcal{C}\cos 2\theta + \mathcal{D}\cos 3\theta + \cdots,$$

with

$$\begin{aligned} \mathcal{A} &= 1 + s^2\alpha^2 + \left(\frac{s(s+1)}{2!}\right)^2 \alpha^4 + \left(\frac{s(s+1)(s+2)}{3!}\right)^2 \alpha^6 + \cdots \\ \mathcal{B}/2 &= s\alpha + s\frac{s(s+1)}{2!}\alpha^3 + \frac{s(s+1)}{2!}\frac{s(s+1)(s+2)}{3!}\alpha^5 + \cdots \\ \mathcal{C}/2 &= \frac{s(s+1)}{2!}\alpha^2 + s\frac{s(s+1)(s+2)}{3!}\alpha^4 + \frac{s(s+1)}{2}\frac{s(s+1)(s+2)(s+3)}{4!}\alpha^6 + \cdots, \end{aligned}$$

and so on. Lagrange added that, after having determined $\mathcal{A}$ and $\mathcal{B}$, the other coefficients could be easily determined by taking the logarithmic derivatives of the preceding relation, cross-multiplying both members and comparing their terms. He wrote that one obtained "as Mr. Euler was the first to find it in his *Recherches sur le mouvement de Saturne*" [27] (p. 25ff.),

$$\mathcal{C} = \frac{(1+\alpha^2)\mathcal{B} - 2s\alpha\mathcal{A}}{(2-s)\alpha}, \quad \mathcal{D} = \frac{2(1+\alpha^2)\mathcal{C} - (1+s)\alpha\mathcal{B}}{(3-s)\alpha}, \quad \mathcal{E} = \frac{3(1+\alpha^2)\mathcal{D} - (2+s))\alpha\mathcal{C}}{(4-s)\alpha}.$$

Lagrange also gave the expression for $\mathcal{E}$. He then deduced from these relations the coefficients of the series $(1 - 2\alpha\cos\theta + \alpha^2)^{-s-1}$ that he needed for his calculations.

The expressions given by Laplace for $b_s^{(0)}$ and $b_s^{(1)}$ are the same as those of $\mathcal{A}$ and $\mathcal{B}$ due to Lagrange. The recurrence relation of Laplace for the coefficients $b_s^{(i)}$ have to be compared with those due to Euler and reproduced by Lagrange, for the first four of them. In [27], Euler considered the expansion of functions of the form $2(1 - 2a\cos\theta + a^2)^{-s}$, with $0 < a < 1$, into series involving the cosines of multiples of θ. Thus (see [28]),

$$2(1 - 2a\cos\theta + a^2)^{-s} = \sum_{j=\infty}^{\infty} b_s^{(j)} \cos j\theta, \quad b_s^{(-j)} = b_s^{(j)},$$

where the coefficients are given by

$$b_s^{(j)} = \frac{2}{\pi}\int_0^\pi \frac{\cos j\theta}{(1 - 2a\cos\theta + a^2)^s}\, d\theta.$$

5. Developments

Let us now express the coefficients $b_s^{(i)}$ for $s = 0, \ldots, 4$.

The coefficients $b_s^{(0)}$ and $b_s^{(1)}$ were expressed by Laplace as the series (8) and (9), respectively, but he did not give their sums for arbitrary values of s. This is what we will now obtain.

For $s = 0$, we have $b_0^{(0)} = 2$ and $b_0^{(1)} = 0$. Using the recurrence relation (2), we obtain that, for all $i \geq 1$, $b_0^{(i)} = 0$.

For $s = 1$, the relation (2) becomes

$$\alpha b_1^{(i)} = (1+\alpha^2) b_1^{(i-1)} - \alpha b_1^{(i-2)},$$

and the series (8) and (9) furnish

$$b_1^{(0)} = 2(1+\alpha^2+\alpha^4+\cdots) = 2/(1-\alpha^2), \quad b_1^{(1)} = 2\alpha(1+\alpha^2+\alpha^4+\cdots) = 2\alpha/(1-\alpha^2).$$

A proof by induction shows us that, for all i,

$$b_1^{(i)} = \frac{2\alpha^i}{1-\alpha^2}. \tag{10}$$

Thus, when s is an integer, all the coefficients $b_{s+1}^{(i)}$ can be computed from the values of $b_s^{(i)}$ using (7). For example, for $s=1$, this relation gives

$$b_2^{(i)} = \frac{2\alpha^i}{(1-\alpha^2)^2}\left[i+\frac{1+\alpha^2}{1-\alpha^2}\right]. \tag{11}$$

This expression of $b_2^{(i)}$ was validated, after some calculations, by plugging it into (2), and it followed that we obtained the sums of the series (8) and (9) for $s=2$, which are

$$b_2^{(0)} = \frac{2(1+\alpha^2)}{(1-\alpha^2)^3}, \quad b_2^{(1)} = \frac{4\alpha}{(1-\alpha^2)^3}.$$

It follows from (9) and (11) that

$$b_2^{(i)} = \frac{b_1^{(i)}}{1-\alpha^2}\left[i+\frac{1+\alpha^2}{1-\alpha^2}\right].$$

Similarly, we obtain

$$b_3^{(i)} = \frac{\alpha^i}{(1-\alpha^2)^3}\left\{\left[i+\frac{1+\alpha^2}{1-\alpha^2}\right]\left[i+2\frac{1+\alpha^2}{1-\alpha^2}\right]+\frac{4\alpha^2}{(1-\alpha^2)^2}\right\}, \tag{12}$$

a formula that was also checked by inserting it into (2), and which gives

$$b_3^{(0)} = \frac{2}{(1-\alpha^2)^5}(1+4\alpha^2+\alpha^4), \quad b_3^{(1)} = \frac{2}{(1-\alpha^2)^5}(3+\alpha^2).$$

From (11) and (12), we obtain

$$b_3^{(i)} = \frac{b_2^{(i)}}{2}\left[i+2\frac{1+\alpha^2}{1-\alpha^2}\right]+\frac{4\alpha^{i+2}}{(1-\alpha^2)^5}.$$

We also computed the coefficients $b_4^{(i)}$ but their expression does not simplify easily. We have

$$b_4^{(i)} = \frac{\alpha^i}{(1-\alpha^2)^5}(-3i\alpha^4+i^2\alpha^4+2\alpha^4+8\alpha^2-2i^2\alpha^2+i^2+3i+2),$$

which leads to

$$b_4^{(0)} = \frac{2\alpha^i}{(1-\alpha^2)^5}(1+4\alpha^2+\alpha^4), \quad b_4^{(1)} = \frac{12\alpha^i(1+\alpha^2)}{(1-\alpha^2)^5}.$$

For larger values of s, the expressions of the coefficients $b_s^{(n)}$ become more complicated, but for all s they have α^n in factor. This result is exploited in the next section.

6. Exploitation

Let us now extract some new consequences from the results given by Euler, Lagrange and Laplace and from those of the preceding sections.

Remember that the Chebyshev polynomials of the first kind are defined by $T_n(x) = \cos(n\theta)$ and those of the second kind by $U_n(x) = \sin((n+1)\theta)/\sin\theta$, where $\theta = \arccos x$.

There exist several generating functions for the Chebyshev polynomials. Among them, we consider the following ones for those of the first kind:

$$\frac{1-\alpha^2}{1-2\alpha x+\alpha^2} = 1 + 2\sum_{n=1}^{\infty} \alpha^n T_n(x), \quad \frac{x-\alpha}{1-2\alpha x+\alpha^2} = \sum_{n=0}^{\infty} \alpha^n T_{n+1}(x), \tag{13}$$

and, for those of the second kind, only this one:

$$\frac{1}{1-2\alpha x+\alpha^2} = \sum_{n=0}^{\infty} \alpha^n U_n(x). \tag{14}$$

We begin by showing how the Fourier expansion (1) given by Laplace can be written as a series in the Chebyshev polynomials of the first kind. With $\theta = \arccos x$, this expansion becomes the Chebyshev series

$$(1-2\alpha\cos\theta+\alpha^2)^{-s} = (1-2\alpha x+\alpha^2)^{-s} = b_s^{(0)} T_0(x)/2 + b_s^{(1)} T_1(x) + b_s^{(2)} T_2(x) + b_s^{(3)} T_3(x) + \cdots \tag{15}$$

This formula provides a whole family of new expansions of the function $(1-2\alpha x+\alpha^2)^{-s}$ in the Chebyshev polynomials of the first kind, a result that seems to have never been given before. Moreover, since the coefficients $b_s^{(n)}$ of the polynomials T_n all have α^n in factor, these expansions provide a whole family of new generating functions for these polynomials. Let us consider those corresponding to $s = 0, 1$ and 2.

For $s = 0$, the expansion (15) gives us

$$(1-2\alpha\cos\theta+\alpha^2)^{-0} = (b_0^{(0)}/2) T_0(x) = 1,$$

which is correct, since $T_0(x) = 1$.

From what precedes, from (10) and from (15), we obtain, for $s = 1$, the following generating function for the Chebyshev polynomials T_n of the first kind:

$$\frac{1}{1-2\alpha x+\alpha^2} = \frac{2}{1-\alpha^2}\left[\frac{T_0(x)}{2} + \sum_{n=1}^{\infty} \alpha^n T_n(x)\right], \tag{16}$$

which is the first formula in (13).

Replacing $1/(1-\alpha^2)$ in (16) by $1+\alpha^2+\alpha^4+\cdots$, gathering the polynomial coefficients of each α^n and using the sum formulæ,

$$U_n(x) = \begin{cases} 2\sum_{j \text{ odd}}^{n} T_j(x), & n \text{ odd} \\ 2\sum_{j \text{ even}}^{n} T_j(x), & n \text{ even} \end{cases}$$

allows us to recover the generating function of the polynomials U_n given in (14).

For $s = 2$, we obtain from (11)

$$\begin{aligned} \frac{1}{(1-2\alpha x+\alpha^2)^2} &= \frac{2}{(1-\alpha^2)^2}\left[\frac{1+\alpha^2}{1-\alpha^2}\frac{T_0(x)}{2} + \sum_{n=1}^{\infty} \alpha^n \left(n + \frac{1+\alpha^2}{1-\alpha^2}\right) T_n(x)\right] \\ &= \frac{1+\alpha^2}{(1-\alpha^2)^2(1-2\alpha x+\alpha^2)} + \frac{2}{(1-\alpha^2)^2}\sum_{n=1}^{\infty} n\alpha^n T_n(x), \end{aligned} \tag{17}$$

which leads to the following new generating function for the Chebyshev polynomials of the first kind:

$$\frac{\alpha^2 x + x - 2\alpha}{(1 - 2\alpha x + \alpha^2)^2} = \sum_{n=1}^{\infty} n\alpha^{n-1} T_n(x), \tag{18}$$

which has to be compared with the second formula in (13).

For larger values of s, we also obtain new, but more complicated, generating functions for the Chebyshev polynomials, since, as seen above, for all s the coefficients $b_s^{(n)}$ have α^n in factor but also include extra terms in n.

Remark 1. *Notice that, setting $z = (1 + \alpha^2)/\alpha$, the recurrence relation (2) for the $b_1^{(i)}$ becomes*

$$b_1^{(i)} = z b_1^{(i-1)} - b_1^{(i-2)},$$

with $b_1^{(0)} = 2/(1 - \alpha^2)$ and $b_1^{(1)} = 2\alpha/(1 - \alpha^2)$. Apart from the initializations and the factor 2 in front of z, this recurrence relation is that of the Chebyshev polynomials, a result mentioned in [29] (Vol. I, pp. 445–446).

Remember now that the generating function for the Gegenbauer polynomials $C_n^{(s)}$ is

$$\frac{1}{(1 - 2\alpha x + \alpha^2)^s} = \sum_{n=0}^{\infty} \alpha^n C_n^{(s)}(x), \tag{19}$$

and it also holds that

$$\frac{1 - \alpha x}{(1 - 2\alpha x + \alpha^2)^{s+1}} = \sum_{n=0}^{\infty} \frac{n + 2s}{2s} \alpha^n C_n^{(s)}(x). \tag{20}$$

For $s = 1/2$, the polynomials $C_n^{(1/2)}$ are the Legendre polynomials P_n and, when $s = 1$, $C_n^{(1)} = U_n$.

From (17) and (19) with $s = 2$, we obtain the following new relation between the generating functions of the polynomials $C_n^{(2)}$ and T_n:

$$\alpha^2(\alpha^2 - 3) \sum_{n=0}^{\infty} \alpha^n C_n^{(2)}(x) = 2 \sum_{n=1}^{\infty} n\alpha^n T_n(x).$$

7. Conclusions

In this Note, we showed that the Fourier expansion of $(1 - 2\alpha\cos\theta + \alpha^2)^{-s}$ studied by Euler, Lagrange and Laplace gives rise to a family of new generating functions for the Chebyshev polynomials. Other results of the same type can be found, for example, in [30] and [31]. Notice also that (14), (15), (19) and (20) furnish four expansions of the function $(1 - 2\alpha x + \alpha^2)^{-s}$, two in Chebyshev polynomials and two in Gegenbauer polynomials.

The moral of this story is *Study the past if you would define the future* (Confucius).

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